$$\frac{d^2\varphi}{dt^2} = \frac{5g}{7(R-r)}\cos\varphi = \frac{\frac{5}{7}g}{l}\cos\varphi, \tag{6}$$

where l is put for $R-r$. Integrating once, and reducing, we find

$$dt = \sqrt{\left(\frac{l}{\frac{10}{7}g}\right)} \cdot \frac{d\varphi}{\sqrt{(\sin\varphi)}}.$$

If we put $l\sin\varphi = x$, we have

$$d\varphi = \frac{dx}{\sqrt{(l^2-x^2)}}; \therefore dt = \frac{l}{\sqrt{(\frac{10}{7}g)}} \cdot \frac{dx}{\sqrt{[x(l^2-x^2)]}}.$$

This value of dt is (see Brande's Encyc., art. pend.) the differential of the time of a semi-oscillation of a pendulum whose length is l, through an arc $= \pi$, or a semi-circumference; the accelerating force being $\frac{5}{7}g$.

We are under obligations to Mr. Adcock and Mr. Siverly for calling our attention to the errors above alluded to. The above corrections being introduced, this solution agrees with Mr. Adcock's solution, he writes, "by the principle that twice the work of gravity equals *vis viva* due to translation plus that due to rotation."

"If t' is the time of a simple pendulum through the same arc φ, $t = t'\sqrt{\frac{7}{5}}$; length of pendulum $= R-r$."

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

Solutions of problems in No. 4 have been received as follows:

From R. J. Adcock, 271, 272; G. M. Day, 275; Geo. Eastwood, 267; Prof. Edgar Frisby, 266; H. Heaton, 266, 271; W. E. Heal, 270, 274; Prof. A. Hall, 275; Chas. H. Kummell, 270, 271, 273, 274, 275; Prof. D. J. McAdam, 269, 270, 274, 275; Artemas Martin, 271; P. Richardson, 270; E. B. Seitz, 267, 269, 270, 271, 272, 273, 274, 275.

We have also received from Prof. W. P. Casey, of San Francisco, Cal., a very elaborate and elegant solution of problem 85, and a geometrical construction of problem 126; also a very simple construction of problem 234; all of which we would be pleased to publish if our space permitted.

At the time of the publication of problem 85, we received several solutions of it, but as all the solutions that have yet been received, except for particular cases, involve equations of a higher degree than the second, we think the quadrilateral has not yet been "constructed". Nevertheless, as the solution by Prof. Casey, above alluded to, is perhaps the most complete and elegant that has been received, it will be inserted as soon as our space will permit; and we hope also to be able, at some future time, to present the geometrical construction of 126, above referred to.

266. "Sum the series

$$\cos 2\theta \cos \theta + \frac{\cos^2 2\theta \cos 3\theta}{2\,.\,3} + \frac{1\,.\,3}{2\,.\,4}\cos^3 2\theta\,\frac{\cos 5\theta}{5} + \ldots \text{ to infinity.''}$$

SOLUTION BY PROF. EDGAR FRISBY, NAVAL OBSERVATORY, WASH., D. C.

$$\text{Let} \qquad x^{\pm n} = e^{\pm n\theta\sqrt{-1}} = \cos n\theta \pm \sqrt{(-1)}\sin n\theta, \qquad (1)$$

$$x^n + x^{-n} = 2\cos n\theta. \qquad (2)$$

$$2S = \cos 2\theta(x+x^{-1}) + \tfrac{1}{2}\cos^2 2\theta\left(\frac{x^3+x^{-3}}{3}\right) + \tfrac{1}{2}.\tfrac{3}{4}\cos^3 2\theta\left(\frac{x^5+x^{-5}}{5}\right) + \&c.$$

$$= \sqrt{(\cos 2\theta)}\left\{\begin{array}{l} x\ \cos^{\frac{1}{2}}2\theta + \tfrac{1}{2}x^3\ .\tfrac{1}{3}(\cos^{\frac{3}{2}}2\theta + \tfrac{1}{2}.\tfrac{3}{4}x^5\ .\tfrac{1}{5}(\cos^{\frac{5}{2}}2\theta + \&c. \\ +x^{-1}\cos^{\frac{1}{2}}2\theta + \tfrac{1}{2}x^{-3}.\tfrac{1}{3}(\cos^{\frac{3}{2}}2\theta + \tfrac{1}{2}.\tfrac{3}{4}x^{-5}.\tfrac{1}{5}(\cos^{\frac{5}{2}}2\theta + \&c. \end{array}\right\}$$

$$= \sqrt{(\cos 2\theta)}.[\sin^{-1} x\sqrt{(\cos 2\theta)} + \sin^{-1} x^{-1}\sqrt{(\cos 2\theta)}]$$

$$= \sqrt{(\cos 2\theta)}.\sin^{-1}[x\sqrt{(\cos 2\theta)}\sqrt{(1-x^{-2}\cos 2\theta)} + x^{-1}\sqrt{(\cos 2\theta)} \times\sqrt{(1-x^2\cos 2\theta)}]$$

$$= \sqrt{(\cos 2\theta)}.\sin^{-1}\sqrt{(\cos 2\theta)}\,[\sqrt{(x^2-\cos 2\theta)} + \sqrt{(x^{-2}-\cos 2\theta)}]$$

$$= \sqrt{(\cos 2\theta)}.\sin^{-1}\sqrt{(\cos 2\theta)}\,[\sqrt{(\sin 2\theta\,\sqrt{-1}} + \sqrt{(-\sin 2\theta\,\sqrt{-1})}], \text{ by (1).}$$

But $\sqrt{(\pm\sin 2\theta\sqrt{-1})} = \sqrt{(\tfrac{1}{2}\sin 2\theta)} \pm \sqrt{(-\tfrac{1}{2}\sin 2\theta)}$,

$$\therefore 2S = \sqrt{(\cos 2\theta)}.\sin^{-1}\sqrt{(\cos 2\theta)}\sqrt{(2\sin 2\theta)} = \sqrt{(\cos 2\theta)}\,.\sin^{-1}\sqrt{(\sin 4\theta)},$$

$$S = \tfrac{1}{2}\sqrt{(\cos 2\theta)}\,.\sin^{-1}\sqrt{(\sin 4\theta)}.$$

[Mr. Heaton, by a similar process, obtains the same result under a slightly different form.]

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267. "What was the duration of a building and loan association in which, for the first eight years, money was loaned at an average premium of $45 per share (of $200), interest paid being 6 per cent."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let a = the value of a share, n = the number of shares, nd = the amt. of dues paid at the end of each interval of time t, p = the premium paid for a loan on a share, r = the rate of interest, and x = the required time in years. Then, the interest on the loans being payable monthly, we have

$$nd(1+\tfrac{1}{12}r)^{12(x-t)},\ nd(1+\tfrac{1}{12}r)^{12(x-2t)},\ nd(1+\tfrac{1}{12}r)^{12(x-3t)},\ \ldots.$$

for the compound amounts of the successive payments of dues; and the sum of these amounts must be equal to $n(a-p)$. Hence, since the amounts form a geometrical series whose least term is nd, and ratio, $(1+\tfrac{1}{12}r)^{12t}$, we have

$$\frac{nd(1+\frac{1}{12}r)^{12x}-nd}{(1+\frac{1}{12}r)^{12t}-1}=n(a-p),$$

whence we find

$$x=\frac{\log\{(a-p)\left[(1+\frac{1}{12}r)^{12t}-1\right]+d\}-\log d}{12\log(1+\frac{1}{12}r)}.$$

In the question we have $a=\$200$, $p=\$45$, and $r=.06$.

In these associations the dues are generally payable weekly, and the amt. due on a share is $\frac{1}{48}$ of the yearly interest on the same; therefore $t=\frac{1}{52}$ yr., and $d=\$\frac{1}{4}$. $\therefore x=9.015$ years.

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268. [No solution received.]

269. "Show that the equations

$$x=(r_1-r_2)\cos\varphi+mr_2\cos\left(\frac{r_1-r_2}{r_2}\varphi\right),$$

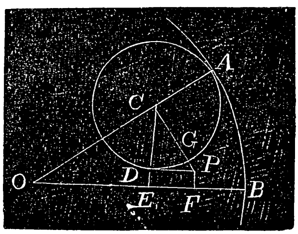
$$y=(r_1-r_2)\sin\varphi-mr_2\sin\left(\frac{r_1-r_2}{r_2}\varphi\right),$$

represent the prolate and curtate hypocycloids; and also that when $r_1=2r_2$ the curve becomes the ellipse

$$\frac{x^2}{(1+m)^2r_2^2}+\frac{y}{(1-m)^2r_2^2}=1."$$

SOLUTION BY PROF. D. J. MCADAM, WASHINGTON, PA.

Take for axis of x the position of the common diameter of the two circles which passes through the generating point P, taken in the figure without the rolling circle.



Let $CA=r_2$, $OA=r_1$, $CP=mr_2$; $COB=\varphi$. Then $OF=x$, $OE+EF=OC\cos COB+CP\cos CPD=(r_1-r_2)\cos\varphi+mr_2\cos CPD$.

But $AG=AB=r_1\varphi$, $\therefore \angle ACG=r_1\varphi\div r_2$, and $ACG=CPD+\varphi$, $\therefore \angle CPD=r_1\varphi\div r_2-\varphi=\frac{r_1-r_2}{r_2}\varphi$; $\therefore x=(r_1-r_2)\cos\varphi+mr_2\cos\left(\frac{r_1-r_2}{r_2}\varphi\right)$,

and $CE-CD=y=(r_1-r_2)\sin\varphi-mr_2\sin\left(\frac{r_1-r_2}{r_2}\varphi\right)$.

Making $r_1=2r_2$, $x=r_2\cos\varphi+mr_2\cos\varphi$; $y=r_2\sin\varphi-mr_2\sin\varphi$.

$$\therefore \cos\varphi=\frac{x}{r_2(1+m)}\ (1);\qquad \sin\varphi=\frac{y}{r_2(1-m)}.\qquad (2)$$

Squaring (1) and (2) and adding,

$$\frac{x^2}{r_2^2(1+m)^2}+\frac{y^2}{r_2^2(1-m)^2}=\sin^2\varphi+\cos^2\varphi=1.$$

270. "Find the locus of the centres of circles tangent to a parabola and to the tangent at its vertex."

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURV., DETROIT, MICH.

Let $$y'^2 = px' \tag{1}$$
be the equation of the parabola, origin of coordinates at vertex. We obtain any point of the required locus if at the point (x', y') of the parabola a normal
$$y''-y' = -\frac{2y'}{p}(x''-x') \tag{2}$$
is drawn and prolonged until its terminating point is as distant from the parabola as from the tangent at the vertex, and this distance is equal to the abscissa x of the required locus. We have then
$$x = \sqrt{[(x'-x)^2+(y'-y)^2]} = x'-x\cos[\tan^{-1}(2y'\div p)]$$
$$= x'-\frac{px}{\sqrt{(p^2+4y'^2)}}, \tag{3}$$
or since $y' = \sqrt{(px')}$, by (1)
$$x = \sqrt{[(x'-x)^2+(\sqrt{px'}-y)^2]} = x'-\frac{x\sqrt{p}}{\sqrt{(p+4x')}}. \tag{4}$$
From the latter of these two equations, by transposing and squaring we find
$$(x'-x)^2(p+4x') = px^2;$$
$$\therefore\quad 4x'^2+(p-8x)x' = 2x(p-2x), \text{ and solving,}$$
$$x' = x-\tfrac{1}{8}p\pm\sqrt{[(x-\tfrac{1}{8}p)^2+\tfrac{1}{2}x(p-2x)]}$$
$$= x-\tfrac{1}{8}p\pm\tfrac{1}{2}\sqrt{[p(x+\tfrac{1}{16}p)]}.$$
Substituting this in the first of (4) we obtain
$$y = \sqrt{p\left\{x-\tfrac{1}{8}p\pm\tfrac{1}{2}\sqrt{[p(x+\tfrac{1}{16}p)]}\right\}} + \sqrt{x^2-\left\{-\tfrac{1}{8}p\pm\tfrac{1}{2}\sqrt{[p(x+\tfrac{1}{16}p)]}\right\}^2}$$
The double sign refers to two loci, the external and internal.

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271. "If three points be taken at random in the circumference of a circle, required the probability that the triangle formed by joining them will be acute."

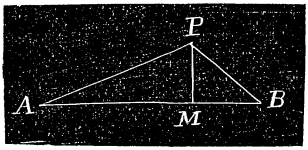
SOLUTION BY HENRY HEATON, ATLANTIC, IOWA.

Let θ be the angle between the diameters drawn from two of the random points. Then, in order that the triangle be acute, the third point must fall upon the arc between the opposite extremities of these diameters. The probability of this is $\theta\div 2\pi$. The probability of θ having any particular value less than π is $d\theta\div\pi$. Hence the required probability is
$$\frac{1}{2\pi^2}\int_0^\pi \theta d\theta = \tfrac{1}{4}.$$

272. "Two lines rotate, uniformly, in opposite directions about two fixed points, the velocity of one being n times that of the other; find the rectangular equation of their intersection."

SOLUTION BY E. B. SEITZ.

Let AP and BP be the two lines rotating about the fixed points A and B, the lines initially being coincident with AB.



Draw PM perpendicular to AD Let $AM = x$, $PM = y$, $AB = a$, $\angle PAM = \theta$, and $\angle PBM = n\theta$. Then we have $\tan\theta = y \div x$, and $\tan n\theta = y \div (a-x)$. But

$$\tan n\theta = \frac{n\tan\theta - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\tan^5\theta - ..}{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4\theta - ..}$$

$$\therefore y\left[1 - \frac{n(n-1)}{1.2}\left(\frac{y}{x}\right)^2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\left(\frac{y}{x}\right)^4 - \ldots\right]$$

$$= (a-x)\left[n\left(\frac{y}{x}\right) - \frac{n(n-1)(n-2)}{1.2.3}\left(\frac{y}{x}\right)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\left(\frac{y}{x}\right)^5 - \ldots\right],$$ the required equation.

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273. "If m and n be the masses of the earth and moon, a the distance between their centers, r the radius of the earth, and if a body fall toward the earth from the point of equal attraction in the line joining their centers, find the time of falling from the hight h to the earth's surface."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO*.

Let E be the center of the earth, M that of the moon, P the point of eq'l attraction, N the position of the body at any time during its motion toward the earth; and let $EM = a$, $EN = x$, $t =$ the time in seconds of falling from P to N, $v =$ the velocity at N, and $t_1 =$ the required time. Then

$$EP = \frac{a\sqrt{m}}{\sqrt{m}+\sqrt{n}}, \quad g' = \frac{gr^2}{x^2} - \frac{gnr^2}{m(a-x)^2}$$

= force acting on the body at N toward the center of the earth,

$$vdv = -g'dx, \qquad dt = -\frac{dx}{v}. \qquad (1, 2)$$

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*We have just learned that Mr. Seitz has accepted the position of Professor of Mathematics in the State Normal School, at Kirksville, Mo., and that after Aug. 28th his address will be Kirksville, Mo.—Ed.

Substituting the value of g' in (1), integrating, and observing that when $x = a\sqrt{m} \div (\sqrt{m}+\sqrt{n})$, $v=0$, we have

$$v^2 = \frac{2gr^2[a\sqrt{m} - (\sqrt{m}+\sqrt{n})x]^2}{am(ax-x^2)}. \tag{3}$$

Substituting the value of v in (2), and integrating, we have

$$t_1 = \left(\frac{am}{2gr^2}\right)^{\frac{1}{2}} \int_r^{h+r} \frac{\sqrt{(ax-x^2)}dx}{a\sqrt{m} - (\sqrt{m}+\sqrt{n})x}.$$

Put $x = \dfrac{ay^2}{1+y^2}$; then $\sqrt{(ax-x^2)} = \dfrac{ay}{1+y^2}$, $a\sqrt{m} - (\sqrt{m}+\sqrt{n})x = \dfrac{a(\sqrt{m}-y^2\sqrt{n})}{1+y^2}$, $dx = \dfrac{2aydy}{(1+y^2)^2}$, $y = \sqrt{\left(\dfrac{r}{a-r}\right)} = y_1$, when $x=r$, and $y = \sqrt{\left(\dfrac{h+r}{a-h-r}\right)} = y_2$, when $x = h+r$. Therefore

$$t_1 = \frac{2a}{r}\left(\frac{a\,m}{2g}\right)^{\frac{1}{2}} \int_{y_1}^{y_2} \frac{y^2dy}{(1+y^2)^2(\sqrt{m}-y^2\sqrt{n})} = \frac{\sqrt{(am)}}{r\sqrt{(2g)}(\sqrt{m}+\sqrt{n})}$$

$$\times \left\{ \sqrt{(ar-r^2)} - \sqrt{[a(h+r)-(h+r)^2]} + a\left(\frac{\sqrt{m}-\sqrt{n}}{\sqrt{m}+\sqrt{n}}\right)\sin^{-1}\sqrt{\left(\frac{h+r}{a}\right)} \right.$$

$$-a\left(\frac{\sqrt{m}-\sqrt{n}}{\sqrt{m}+\sqrt{n}}\right)\sin^{-1}\sqrt{\left(\frac{r}{a}\right)} + \frac{a\,{}^4\!\sqrt{(m\,n)}}{\sqrt{m}+\sqrt{n}}\log\left[\frac{\sqrt{(a-h-r)}\,{}^4\!\sqrt{m}+\sqrt{(h+r)}\,{}^4\!\sqrt{n}}{\sqrt{(a-h-r)}\,{}^4\!\sqrt{m}-\sqrt{(h+r)}\,{}^4\!\sqrt{n}}\right]$$

$$\left. -\frac{a\,{}^4\!\sqrt{(m\,n)}}{\sqrt{m}+\sqrt{n}}\log\left[\frac{\sqrt{(a-r)}\,{}^4\!\sqrt{m}+\sqrt{r}\,{}^4\!\sqrt{n}}{\sqrt{(a-r)}\,{}^4\!\sqrt{m}-\sqrt{r}\,{}^4\!\sqrt{n}}\right] \right\}$$

COR.—If $n=0$, and $h = a-r$, we have

$$t_1 = \sqrt{[a \div (2gr^2)]}[\sqrt{(ar-r^2)} + \tfrac{1}{2}\pi a - a\sin^{-1}\sqrt{(r \div a)}],$$

the usual formula for the time of a body falling from a great distance.

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274. "Required the shortest distance between two curves whose eq's are

$$4x^2+9y^2-144 = 0,$$
$$x'^2+y'^2-26x'-32y'+25 = 0."$$

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY.

The first curve is an ellipse, the semi axes of which are $a=6$ and $b=4$, the center being at the origin and the major axis coinciding with the x-axis. The shortest distance between this ellipse and any other curve must always be measured on a normal. Now the equation of the normal to (1) at the point (x, y) is

$$y''-y = \frac{36y}{16x}\left(x''-x\right). \tag{3}$$

Equation (2) is that of a circle, radius 20 and $x'_0 = 13$; $y'_0 = 16$ as co-ordinates of the center. The shortest line between the ellipse and circle

must pass through the center. Causing the normal (3) to pass through the center of the circle, we obtain the equation of that normal on which the shortest distance is measured, viz.;

$$y'' = 16 + \frac{36y}{16x}\left(x''-13\right). \tag{4}$$

To find the intersection of this line with the ellipse, we have, substituting $\frac{x}{y}$ for $\frac{x''}{y''}$: $16x(y-16) = 36y(x-13)$, $\therefore\ y = \dfrac{64x}{117-5x}$. (5)

By (1) we have $$y = \tfrac{2}{3}\sqrt{(36-x^2)}; \tag{6}$$

$$\therefore\ \frac{64x}{117-5x} = \tfrac{2}{3}\sqrt{(36-x^2)},\ \text{or } 25x^4-1170x^3+22005x^2+42120x-492804 = 0.$$

The real roots of this eq'n are $x_1 = +4.238395$; $x_2 = -4.96895$.

The latter evidently belongs to the maximum distance and is not considered. By (5) or (6) we obtain $y_1 = +2.831258$.

We have now, for the shortest distance,

$$d_0 = \sqrt{[(x'_0-x_1)^2+(y'_0-y_1)^2]}-r',$$

where r' = radius of circle = 20. We have then, in numbers,

$$d_0 = 15.817126-20 = -4.182874.$$

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275. "Find the moments of inertia of an elliptical disk: (1), about a right line in the plane of the disk and parallel to the axis of x: (2), about a right line parallel to the axis of y, the equation of the disk being

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0."$$

SOLUTION BY PROF. HALL.

Solving the equation for x, we have

$$x = -\frac{by+d}{a} \pm \frac{1}{a}\sqrt{\left[(b^2-ac)y^2+2(bd-ae)y+d^2-af\right]}.$$

If k be the mass of a unit of area, and q be the distance of the right line from the axis of x, the moment of inertia is

$$-\frac{2k}{a}\int_{y_1}^{y_2}(y-q)^2\sqrt{[(b^2-ac)y^2+2(bd-ae)y+d^2-af]}.dy.$$

The values of y_1 and y_2 for the limits of integration are found by putting the quantity under the radical equal to zero. The integrals involved are all of known forms, but the reductions are complicated, and it is better to proceed as follows.

Let M be the mass of the disk, and if $2A$ and $2B$ are the principal axes of the disk we know that the moments of inertia about these axes are

$$\tfrac{1}{4}M\,.\,B^2: \text{ and } \tfrac{1}{4}M\,.\,A^2.$$

We have therefore only to resolve these moments along the direction of the given right line and add the quantities,

$$Mq^2: \qquad \text{and } Mp^2,$$

q and p being now the distances of the right line from centre of the ellipse.

If the equation be transferred to the centre of the ellipse it becomes

$$ax^2+2bxy+cy^2+f'=0,$$

whence $$-f'=\frac{b^2f+d^2c+e^2a-2bde-acf}{ac-b^2}=\frac{\Delta}{ac-b^2}$$

If θ be the angle that the axis of x makes with the major axis of the ellipse, and if we put $N=\sqrt{[(a-c)^2+4b^2]}$, we have

$$\cos\theta^2=\frac{N+(a-c)}{2N}: \qquad \text{and } \sin\theta^2=\frac{N-(a-c)}{2N}.$$

We have also for the semi-axes

$$A^2=\frac{\Delta}{ac-b^2}\cdot\frac{2}{a+c+N}: \qquad B^2=\frac{\Delta}{ac-b^2}\cdot\frac{2}{a+c-N}.$$

The moment of inertia about a line parallel to the axis of x and passing through the centre of the ellipse is $\tfrac{1}{4}M.(B^2\cos\theta^2+A^2\sin\theta^2)$, which becomes on reduction

$$\frac{M}{4}\cdot\frac{a\Delta}{(ac-b^2)^2}: \qquad \text{and the required moment is}$$

$$\frac{M}{4}\cdot\frac{a\Delta}{(ac-b^2)^2}+Mq^2.$$

For the moment about a line parallel to the axis of y we have, since the equation of the disk is symmetrical,

$$\frac{M}{4}\cdot\frac{a\Delta}{(ac-b^2)^2}+Mp^2.$$

PROBLEMS.

276. By *O. L. Mathies, Reistertown, Md.*—Given the chord AC of a circle, the side AB of a right angled triangle constructed on AC as hypothenuse, and the length of a perpendicular from A upon the line joining the right angle at B with the centre of the circle; to find the radius of the circle.

277. By *W. E. Heal, Wheeling, Indiana.*—Solve, algebraically the eq'n

$$x^{17}-1=0.$$